Chapter 1.

Indices.

A tale of two students. Situation One.

Two students are asked to work out $5^{17} \div 5^{15}$ without the help of a calculator.

One of the students starts to work out the powers of 5:

$$5^2 = 5 \times 5 = 25$$
$$5^3 = 5 \times 5 \times 5 = 125$$
$$5^4 = 5 \times 5 \times 5 \times 5 = 625$$
$$5^5 = 5 \times 5 \times 5 \times 5 \times 5 = 3125 \qquad \text{etc.}$$

The other student simply looks at $5^{17} \div 5^{15}$ and says:

"The answer must be 25".

How is the second student able to work out the answer so quickly?

A tale of two students. Situation Two.

Two students are asked to work out how much will be in a savings account after 25 years if \$5000 is invested in the account at the beginning of the 25 years, interest is added at 10% per year and interest added in one year itself earns interest in subsequent years (i.e. *compound interest* is involved).

One of the students starts working on a year by year basis as follows:

Amount in account during year 1 = \$5000

At the end of year 1, interest earned = 10% of \$5000 (i.e. \$500)

Amount in account during year 2 = \$5500 (= \$5000 + \$500)

At the end of year 2, interest earned = 10% of \$5500 (i.e. \$550)

Amount in account during year 3 = \$6050 (= \$5500 + \$550)

At the end of year 3, interest earned = 10% of \$6050 (i.e. \$605)

Amount in account during year 4 = \$6655 etc

The other student does a quick calculation using a calculator and said:

"After 25 years the account will be worth \$54173·53".

How was this second student able to determine the answer so quickly?

Situation Three.

A certain sum of money, \$P, is invested at 8% interest compounded annually.
How many years will it take to become \$(4P) ?

What if the interest rate had been 12% rather than 8% ?

Revision of powers, or indices.

The first situation on the previous page involved a number being raised to some *power*, or *exponent*, and the other two situations could be solved using this idea. As was mentioned in the *Preliminary work* section at the beginning of this book it is anticipated that you are familiar with this idea and are also familiar with *zero and negative integers as powers* and that you may be aware of some of the *index laws.*

Note • In an expression like 2^3, the number 3 is the *index* and the number 2 is the *base.* The index shows the *power* to which the base is raised.

Read through the following to revise, and possibly extend, your understanding of these concepts and then work through the exercise that follows to practice the application of the ideas.

Notice that

$$\begin{aligned} a^2 \times a^3 &= (a \times a) \times (a \times a \times a) &&= a^5 \quad (= a^{2+3}) \\ a^3 \times a^5 &= (a \times a \times a) \times (a \times a \times a \times a \times a) &&= a^8 \quad (= a^{3+5}) \\ a^2 \times a^7 &= (a \times a) \times (a \times a \times a \times a \times a \times a \times a) &&= a^9 \quad (= a^{2+7}) \end{aligned}$$

To generalise:

$$\boxed{\mathbf{a^n \times a^m = a^{n+m}}}$$

Notice that

$$\begin{aligned} a^5 \div a^3 &= \frac{\cancel{a \times a \times a} \times a \times a}{\cancel{a \times a \times a}} &&= a^2 \quad (= a^{5-3}) \\ a^7 \div a^4 &= \frac{\cancel{a \times a \times a \times a} \times a \times a \times a}{\cancel{a \times a \times a \times a}} &&= a^3 \quad (= a^{7-4}) \\ a^8 \div a^3 &= \frac{\cancel{a \times a \times a} \times a \times a \times a \times a \times a}{\cancel{a \times a \times a}} &&= a^5 \quad (= a^{8-3}) \end{aligned}$$

To generalise:

$$\boxed{\mathbf{a^n \div a^m = a^{n-m}}}$$

From the above rule it follows that

$$\begin{aligned} a^5 \div a^5 &= a^0 \\ a^7 \div a^7 &= a^0 \\ a^{12} \div a^{12} &= a^0 \end{aligned}$$

However, $a^5 \div a^5$, $a^7 \div a^7$ and $a^{12} \div a^{12}$ each involve something divided by itself, which must equal 1 (provided the "something" does not equal zero).

Hence:

$$\boxed{\mathbf{a^0 = 1}}$$

Again using the fact that $a^n \div a^m = a^{n-m}$ it follows that

$$\begin{aligned} & a^0 \div a^n = a^{-n} \\ \text{But} \quad & a^0 \div a^n = 1 \div a^n \\ & \qquad\quad\;\; = \frac{1}{a^n} \end{aligned}$$

Hence:

$$\boxed{\mathbf{a^{-n} = \frac{1}{a^n}}}$$

Example 1

Evaluate each of the following without the use of a calculator.

(a) $8^{19} \div 8^{17}$ (b) $3^0 + 2 \times 5^0$ (c) $2^8 \div 8$ (d) $\dfrac{6^3 \times 6^7}{6^8}$

(a) $8^{19} \div 8^{17} = 8^{19-17} = 8^2 = 64$

(b) $3^0 + 2 \times 5^0 = 1 + 2 \times 1 = 1 + 2 = 3$

(c) $2^8 \div 8 = 2^8 \div 2^3 = 2^5 = 32$

(d) $\dfrac{6^3 \times 6^7}{6^8} = 6^{3+7-8} = 6^2 = 36$

Example 2

Express each of the following as a power of 5 (i.e. in the form 5^k).

(a) $5^8 \div 5^5$ (b) $5^7 \times 25$ (c) $\dfrac{1}{5^4}$ (d) 0·2

(a) $5^8 \div 5^5 = 5^{8-5} = 5^3$

(b) $5^7 \times 25 = 5^7 \times 5^2 = 5^9$

(c) $\dfrac{1}{5^4} = 5^{-4}$

(d) $0{\cdot}2 = \dfrac{1}{5} = 5^{-1}$

Using the fact that $a^n \times a^m = a^{n+m}$ it follows that

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1$$

Thus $a^{\frac{1}{2}}$ multiplied by itself gives a. Hence $a^{\frac{1}{2}}$, or $a^{0\cdot5}$, is the square root of a.

Similarly $a^{\frac{1}{3}}$ is the cube root of a, $\sqrt[3]{a}$, $a^{\frac{1}{4}}$ is the fourth root of a, $\sqrt[4]{a}$,

$a^{\frac{1}{5}}$ is the fifth root of a, $\sqrt[5]{a}$, etc.

Hence:

$$\mathbf{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

Note: A power of $\frac{1}{2}$ represents the **positive** square root, just as the radical sign, $\sqrt{\ }$, does. i.e.

If $x = \sqrt{4}$ then $x = 2$.

If $x = 4^{\frac{1}{2}}$ then $x = 2$.

But if $x^2 = 4$ then $x = \pm 2$.

Notice that

$$(a^2)^3 = (a \times a) \times (a \times a) \times (a \times a) = a^6 \quad (= a^{2\times 3})$$
$$(a^4)^2 = (a \times a \times a \times a) \times (a \times a \times a \times a) = a^8 \quad (= a^{4\times 2})$$
$$(a^3)^3 = (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a) = a^9 \quad (= a^{3\times 3})$$

To generalise:

$$\boxed{(a^n)^m = a^{n\times m}}$$

Notice that

$$(ab)^2 = (a \times b) \times (a \times b) = a^2 \times b^2$$
$$(ab)^3 = (a \times b) \times (a \times b) \times (a \times b) = a^3 \times b^3$$
$$(ab)^4 = (a \times b) \times (a \times b) \times (a \times b) \times (a \times b) = a^4 \times b^4$$

To generalise:

$$\boxed{(ab)^n = a^n \times b^n}$$

Notice that

$$\left(\frac{a}{b}\right)^5 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^5}{b^5}$$

To generalise:

$$\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

Example 3

Evaluate each of the following without the use of a calculator.

(a) $16^{0\cdot 5}$ (b) $(2^6 \div 2^4)^2$ (c) $\left(3\frac{3}{8}\right)^{\frac{1}{3}}$ (d) $8^{\frac{4}{3}}$

(a)
$$16^{0\cdot 5} = \sqrt{16}$$
$$= 4$$

(b)
$$(2^6 \div 2^4)^2 = (2^2)^2$$
$$= 2^4$$
$$= 16$$

(c)
$$\left(3\frac{3}{8}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{27}{8}}$$
$$= \frac{\sqrt[3]{27}}{\sqrt[3]{8}}$$
$$= \frac{3}{2} \quad \text{i.e. } 1\cdot 5$$

(d)
$$8^{\frac{4}{3}} = (8^{\frac{1}{3}})^4$$
$$= 2^4$$
$$= 16$$

The previous answers can be confirmed with a calculator.

$16^{0.5}$	4
$(2^6/2^4)^2$	16
$\sqrt[3]{3+\frac{3}{8}}$	1.5
$8^{\frac{4}{3}}$	16

Example 4

Simplify each of the following, expressing your answers in terms of positive indices.

(a) $5a^2y^3 \times 6a^4y^7$ (b) $\dfrac{25ab^9}{15a^7b^4}$ (c) $(a^{-7} \times a^3)^{\frac{1}{2}}$ (d) $\left(\dfrac{a^3 b}{b^3}\right)^{-2}$

(a) $5a^2y^3 \times 6a^4y^7 = 30\,a^{2+4}y^{3+7}$

$= 30\,a^6y^{10}$

(b) $\dfrac{25ab^9}{15a^7b^4} = \dfrac{25}{15} \times a^{1-7}b^{9-4}$

$= \dfrac{5}{3} \times a^{-6}b^5$

$= \dfrac{5b^5}{3a^6}$

(c) $(a^{-7} \times a^3)^{\frac{1}{2}} = (a^{-7+3})^{\frac{1}{2}}$

$= (a^{-4})^{\frac{1}{2}}$

$= a^{-2}$

$= \dfrac{1}{a^2}$

(d) $\left(\dfrac{a^3 b}{b^3}\right)^{-2} = \left(\dfrac{a^3}{b^2}\right)^{-2}$

$= \dfrac{a^{-6}}{b^{-4}}$

$= \dfrac{1}{a^6} \div \dfrac{1}{b^4}$

$= \dfrac{b^4}{a^6}$

Again the same answers can be obtained using the ability of some calculators to simplify expressions.

$5a^2y^3 \times 6a^4y^7$	$30 \cdot a^6 \cdot y^{10}$
$\dfrac{25a \times b^9}{15a^7 \times b^4}$	$\dfrac{5 \cdot b^5}{3 \cdot a^6}$

$(a^{-7} \times a^3)^{\frac{1}{2}}$	$\dfrac{1}{a^2}$
$\left(\dfrac{a^3 b}{b^3}\right)^{-2}$	$\dfrac{b^4}{a^6}$

Exercise 1A

Evaluate each of the following without the use of a calculator.

1.	$6^{20} \div 6^{18}$	2.	$2^{19} \div 2^{16}$	3.	$2^{12} \div 2^{8}$
4.	3^0	5.	5^0	6.	$5^0 + 2^0$
7.	$(5+2)^0$	8.	$\dfrac{2^3 \times 2^5}{2^7}$	9.	$\dfrac{2^3 \times 2^5}{2^8}$
10.	$\dfrac{2^{10}}{2^3 \times 2^4}$	11.	1^5	12.	$(-1)^5$
13.	$(-2)^5$	14.	$(-1)^{60}$	15.	$(-1)^{61}$
16.	$16^{\frac{1}{2}}$	17.	$25^{\frac{1}{2}}$	18.	$8^{\frac{1}{3}}$
19.	$81^{\frac{1}{4}}$	20.	$81^{\frac{1}{2}}$	21.	3^{-2}
22.	4^{-1}	23.	2^{-3}	24.	$2^{-1} + 4^{-1}$
25.	$(2+4)^{-1}$	26.	$9^{\frac{1}{2}} + 16^{\frac{1}{2}}$	27.	$(9+16)^{\frac{1}{2}}$
28.	$25^{-\frac{1}{2}}$	29.	5^0	30.	$(5^0)^2$
31.	$(5^0)^{\frac{1}{2}}$	32.	$(-8)^{\frac{1}{3}}$	33.	$25^{\frac{3}{2}}$
34.	$\left(1\frac{7}{9}\right)^{\frac{1}{2}}$	35.	$\left(2\frac{1}{4}\right)^{-\frac{1}{2}}$	36.	$9^{\frac{3}{2}}$
37.	$9^{-\frac{3}{2}}$	38.	2^{-4}	39.	$5^0 + 2^{-1}$
40.	$125^{\frac{2}{3}}$	41.	$(-125)^{\frac{2}{3}}$	42.	$64^{-\frac{2}{3}}$
43.	$64^{-\frac{3}{2}}$				

Express each of the following as a power of 2 (i.e. in the form 2^k).

44.	$2^7 \times 2^9$	45.	$2^6 \times 2^4 \times 2^3$	46.	$2^9 \div 2^3$
47.	$2^{11} \times 8$	48.	$2^{11} \div 8$	49.	1
50.	$\dfrac{1}{2}$	51.	$\dfrac{1}{2^3}$	52.	$\dfrac{1}{8}$

Express each of the following as a power of 3 (i.e. in the form 3^k).

53. 27
54. 81
55. 1
56. $\frac{1}{3}$
57. $\sqrt{3}$
58. $\sqrt[4]{3}$
59. $\frac{1}{27}$
60. $\frac{1}{\sqrt{3}}$
61. $3 \times 3^2 \times 3^3$

Express each of the following as a power of 10 (i.e. in the form 10^k).

62. 100
63. $\frac{1}{10}$
64. $0{\cdot}1$
65. $\frac{1}{100}$
66. $0{\cdot}01$
67. 1
68. $(10^2)^3$
69. $(10^3)^2$
70. 100^3
71. 1000^3
72. $(0{\cdot}1)^3$
73. $\sqrt{10}$
74. $(\sqrt{10}\ )^6$
75. $\frac{1}{\sqrt{10}}$
76. $\sqrt{10^3}$

Determine the value of n in each of the following.

77. $2 \times 2 \times 2 \times 2 \times 2 = 2^n$
78. $3 \times 3 \times 3 \times 3 = 3^n$
79. $8 = 2^n$
80. $625 = 5^n$
81. $2^3 \times 2^4 = 2^n$
82. $3^8 \div 3^3 = 3^n$
83. $(3^2)^4 = 3^n$
84. $2^3 \times 2^4 \times 2 = 2^n$
85. $2^{15} \div 2^5 = 2^n$
86. $\frac{3^7 \times 3^4}{3^3} = 3^n$
87. $2^7 \times 4 = 2^n$
88. $\frac{2^4 \times 2^{11}}{2^7} = 2^n$
89. $(2^3 \times 2^2)^2 = 2^n$
90. $(2^3)^2 \times 2^2 = 2^n$
91. $2^3 \times (2^2)^2 = 2^n$
92. $6^3 = 2^n 3^n$
93. $\frac{9}{16} = \left(\frac{3}{4}\right)^n$
94. $3^4 \times 3^n \times 3 = 3^7$
95. $\frac{3^n}{3^4} = 3^{11}$
96. $(3^n)^5 = 3^{15}$
97. $(5^2)^n \times 5^3 = 5^{11}$

Simplify each of the following without the assistance of a calculator and expressing your answers in terms of positive integers.
(Then see if you can get the same answer on a calculator that has the ability to simplify expressions like these.)

98. $a^4 \times a^3$

99. $a^2 \div a^5$

100. $b^3 \div b^8$

101. $(b^3)^2$

102. $(b^{-2})^3$

103. $(a^{\frac{1}{2}})^4$

104. $(a^{-3})^2$

105. $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$

106. $(b^2)^3 \times b^4$

107. $(b^2 \times b^4)^3$

108. $-4a^2 \times a^3$

109. $(-4a)^2 \times a^3$

110. $(a^2 \times a^{-3})^2$

111. $\left(\dfrac{a^3 b^2}{b^3}\right)^3$

112. $2a^{-1} \times 3a^4$

113. $\dfrac{a^4 \times a}{a^8}$

114. $\dfrac{a^3 b}{ab}$

115. $\dfrac{2a^4 b}{a^3 b}$

116. $4a^2 \times 2a^3$

117. $4a^2 \times (2a)^3$

118. $\dfrac{8a^3 b^5}{2ab}$

119. $\dfrac{10ay^5}{a^6 y^3}$

120. $\dfrac{12a^2 b^7}{8a^6 b^{10}}$

121. $\dfrac{6xy^2}{18x^2 y}$

122. $\dfrac{(-3a^3 b)^3}{3ab^2}$

123. $\dfrac{(a^2 b^3)^2}{a^2 b}$

124. $\dfrac{(3a^2)^2 \times b}{6b^3}$

125. $\dfrac{(a^4 \times a^{-12})^{\frac{1}{2}}}{a}$

126. $\left(\dfrac{x^4}{y^3}\right)^{-1}$

127. $\left(\dfrac{a^5 b}{ab^3}\right)^{-2}$

128. $\dfrac{a^6 + a^3}{a^2}$

129. $\dfrac{a^7 + a^9}{a^2 \times a^3}$

130. $\dfrac{6a^4 + 9a^3}{3a^3}$

Challenging:

131. $\dfrac{2^{n+2} + 12}{5 \times 2^n + 15}$

132. $\dfrac{2^{2n+3} - (2^n)^2}{2^n}$

133. $\dfrac{3^{n+1} + 9}{3^{n-1} + 1}$

Solving equations involving indices.

Equations involving indices could have the unknown as the power or index, as in the following equations:

$2^x = 16$ $\quad 3 \times 5^x = 375$ $\quad 4^x - 5 = 11$ $\quad 8^x = 4$ $\quad 25^{3x-1} = 0{\cdot}2$

or the unknown could be the base, as in the following:

$x^2 = 36$ $\quad x^{0{\cdot}5} = 9$ $\quad x^{-1} = 9$ $\quad 3 + x^{\frac{1}{2}} = 7$ $\quad \dfrac{x}{\sqrt{x}} = 7$

Questions 77 to 97 in the previous exercise all involved equations in which the unknown, in that case n, featured as an index. The next example shows a few more of this type.

Example 5 (Unknown as the index.)

Solve the following equations (a) $2^x = 16$ (b) $3 \times 5^x = 375$

(c) $4^x - 5 = 11$ (d) $8^x = 4$

(e) $25^{3x-1} = 0{\cdot}2$

(a) Given: $2^x = 16$

An awareness of the powers of 2, i.e. $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, ... enables the equation to be solved mentally: $x = 4$

(b) Given: $3 \times 5^x = 375$

Divide each side by 3: $5^x = 125$

From an awareness of the powers of 5: $x = 3$

(c) Given: $4^x - 5 = 11$

Add 5 to both sides to isolate 4^x $\quad 4^x = 16$

From an awareness of the powers of 4: $x = 2$

(d) Given: $8^x = 4$

You may again be able to determine the answer intuitively but, if not, notice that we can express each side of the equation as powers of the same base, in this case 2:

$$(2^3)^x = 2^2$$

i.e. $2^{3x} = 2^2$

Hence $3x = 2$

$\therefore$ $x = \dfrac{2}{3}$

(e) Given: $25^{3x-1} = 0{\cdot}2$

Noticing that 25 and 0·2 can both be expressed as powers of 5:

$$(5^2)^{3x-1} = 5^{-1}$$

i.e. $5^{6x-2} = 5^{-1}$

Hence $6x - 2 = -1$

$6x = 1$

$\therefore$ $x = \dfrac{1}{6}$

Alternatively the previous equations could be solved using the ability of some calculators to solve equations.

solve($2^x = 16, x$)	$\{x = 4\}$
solve($3\times5^x = 375, x$)	$\{x = 3\}$
solve($4^x - 5 = 11, x$)	$\{x = 2\}$
solve($8^x = 4, x$)	$\left\{x = \frac{2}{3}\right\}$
solve($25^{3x-1} = 0.2, x$)	$\left\{x = \frac{1}{6}\right\}$

If the unknown is the base the technique is to steadily reduce the power of the unknown by performing suitably chosen operations to both sides of the equation, as shown in the next example.

It must be remembered though that when reducing $x^n = c$ to $x = \sqrt[n]{c}$ then if n is even we must say that $x = \pm\sqrt[n]{c}$.

e.g. if $x^2 = 64$ then $x = \pm\sqrt{64} = \pm 8$

but if $x^3 = 64$ then $x = \sqrt[3]{64} = 4$

Example 6 (Unknown as the base.)

Solve the following equations

(a) $x^2 = 36$

(b) $x^{0\cdot5} = 9$

(c) $x^{-1} = 9$

(d) $3 + x^{\frac{1}{2}} = 7$

(e) $\dfrac{x}{\sqrt{x}} = 7$

(a) Given: $x^2 = 36$

Take the square root of each side: $x = \pm\sqrt{36}$

$= \pm 6$

(b) Given: $x^{0\cdot5} = 9$

i.e. $\sqrt{x} = 9$

Square each side: $(x^{0\cdot5})^2 = 9^2$

i.e. $x = 81$

(c) Given: $x^{-1} = 9$

i.e. $\frac{1}{x} = 9$

Multiply each side by x: $1 = 9x$

Divide by 9 to isolate x: $\frac{1}{9} = x$

i.e. $x = \frac{1}{9}$

[Alternatively: Raise each side of the initial equation to the power −1 to give $x = 9^{-1}$

i.e., as before, $x = \frac{1}{9}$]

(d) Given $3 + x^{\frac{1}{2}} = 7$

Subtract 3 from each side: $x^{\frac{1}{2}} = 4$

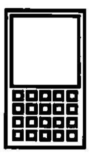
Squaring each side gives: $x = 16$

(e) Given $\frac{x}{\sqrt{x}} = 7$

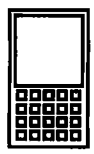
i.e. $x^{1-0{\cdot}5} = 7$

$x^{0{\cdot}5} = 7$

Squaring each side gives: $x = 49$



Again these same equations could be solved using the equation solving ability of some calculators



Exercise 1B

Solve each of the following equations without the help of a calculator.

Unknown in the index.

1. $2^x = 8$	2. $2^x = 32$	3. $2^x = 128$
4. $2^x = \frac{1}{8}$	5. $2^x = \frac{1}{32}$	6. $2^x = \frac{1}{128}$
7. $2^a = \sqrt{2}$	8. $2^a = \frac{1}{\sqrt{2}}$	9. $2^y = \frac{1}{4}$
10. $5^c = 125$	11. $10^d = 1000$	12. $4^x - 3 = 61$

13. $3^x - 2 = 25$
14. $2^y - 6 = 58$
15. $2 \times 5^x = 50$
16. $3^{2x} = 9$
17. $5^{4x} = 125$
18. $5^{4+x} = 125$
19. $3 \times 2^x = 24$
20. $\frac{10^x}{5} = 20$
21. $\frac{2^x}{4} = 8$
22. $16^k = 8$
23. $16^p = \frac{1}{2}$
24. $9^x = 27$
25. $2^{3x+1} = 16$
26. $2^{15-2h} = 8$
27. $4^{x-1} = 0{\cdot}5$
28. $3^{x+1} = \frac{27}{3^x}$
29. $16^{x+2} = 128$
30. $5^{2n-1} = 125$

<u>Unknown in the base.</u>

31. $a^2 = 16$
32. $p^2 = 100$
33. $x^3 = 8$
34. $x^3 = 64$
35. $x^{\frac{1}{2}} = 4$
36. $x^{\frac{1}{3}} = 4$
37. $h^{-1} = 4$
38. $y^{-1} = 2$
39. $p^{-1} = \frac{1}{3}$
40. $x^{0{\cdot}5} = 100$
41. $3x^2 = 75$
42. $9x^2 = 4$
43. $x^4 + 7 = 88$
44. $3 + x^{\frac{1}{3}} = 13$
45. $p^2 - 3 = 13$
46. $\frac{x^5}{x^2} = 64$
47. $\frac{x}{\sqrt{x}} = 9$
48. $\frac{x^2}{x^3} = 16$
49. $(w-2)^3 = 8$
50. $(2x-1)^3 = 27$
51. $(h+1)^{\frac{1}{2}} = 5$
52. $(x-3)^2 = 16$
53. $2w^{\frac{1}{2}} = 3$
54. $2z^{\frac{1}{3}} = 3$

55. (a) Consider the equation $x^5 = 16x^3$

Dividing both sides by x^3 gives $\frac{x^5}{x^3} = \frac{16x^3}{x^3}$

i.e. $x^2 = 16$

and hence $x = \pm 4.$

However, as well as these two solutions of $x = 4$ and $x = -4$, there is another value for x that satisfies the original equation. What is this third solution to the original equation?

(b) Instead of dividing each side of the original equation by x^3 to solve it, subtract $16x^3$ from each side and then factorise and solve.

(c) Solve each of the following equations:

(i) $x^2 = 4x$ (ii) $x^3 = 25x$ (iii) $x^3 = 25x^2$ (iv) $x^3 + 16x = 0$

What if we cannot solve the equation mentally or algebraically?

Asked to solve the equation $8^x = 4$

Our familiarity with powers of 2 allows us to write this as $(2^3)^x = 2^2$

i.e. $2^{3x} = 2^2$

Hence $3x = 2$

and so $x = \frac{2}{3}$

Similarly, given the equation $25^{3x-1} = 0{\cdot}2$

Our familiarity with powers of 5 allows us to write this as $5^{6x-2} = 5^{-1}$

Giving $x = \frac{1}{6}$

However, suppose we were asked to solve $2^x = 11$

or perhaps $5^{2x-3} = 48$?

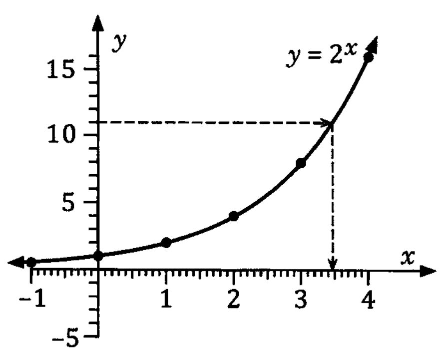
In such cases we could use
- the solve facility of some calculators,
- or a graphical approach,
- or trial and adjustment,
- or logarithms, a concept that will be introduced in a later unit of *Mathematical Methods*.

Using the solve facility of some calculators:

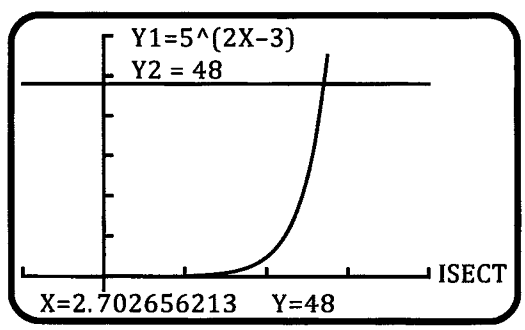
```
solve(2^x = 11, x)
        {x = 3.459431619}
solve(5^(2x-3) = 48, x)
        {x = 2.702656213}
```

A graphical approach:

To solve	$2^x = 11$		To solve	$5^{2x-3} = 48$
draw	$y = 2^x$		draw	$y = 5^{2x-3}$
and see where	$y = 11$		and see where	$y = 48$



$x \approx 3{\cdot}46$



$x = 2{\cdot}703$ (to 3 dp)

Trial and adjustment.

To solve $2^x = 11$

Note that $2^3 = 8$

and $2^4 = 16$

Try $x = 3{\cdot}5$ $\quad 2^{3\cdot5} \approx 11{\cdot}31$ (by calculator)

Try $x = 3{\cdot}4$ $\quad 2^{3\cdot4} \approx 10{\cdot}56$ (by calculator)

Try $x = 3{\cdot}45$ $\quad 2^{3\cdot45} \approx 10{\cdot}93$ (by calculator)

Thus for $2^x = 11$

x must lie between 3·45 and 3·5.

Therefore, correct to one decimal place, $x = 3{\cdot}5$.

To solve $5^{2x-3} = 48$

Note that $5^2 = 25$

and $5^3 = 125$

Try 2·5 $\quad 5^{2\cdot5} \approx 55{\cdot}9$

Try 2·4 $\quad 5^{2\cdot4} \approx 47{\cdot}59$

Try 2·45 $\quad 5^{2\cdot45} \approx 51{\cdot}58$

Thus, correct to 1 decimal place

$$2x - 3 = 2{\cdot}4$$
$$2x = 5{\cdot}4$$
$$x = 2{\cdot}7$$

Exercise 1C

Trial and adjustment.

Solve each of the following equations using "trial and adjustment" and giving answers correct to one decimal place.

1. $2^x = 23$
2. $3^x = 33$
3. $5^x = 50$
4. $7^x + 5 = 245$
5. $2^{x-1} = 51$
6. $3^{x+2} = 100$

Graphical methods.

Solve each of the following equations using the ability of some calculators to graph functions. (Give answers correct to two decimal places.)

7. $3^x = 15$
8. $5^x = 61$
9. $4^x = 100$
10. $2^x + x = 83$
11. $2^x + 3^x = 84$
12. $5^x - 3^x = 250 - x$

The solve facility of some calculators.

Solve each of the following using the solve facility of some calculators. (Give answers correct to two decimal places.)

13. $2^x = 345$
14. $2^x = 0{\cdot}35$
15. $5^x = 1100$
16. $2^{2x-1} = 51$
17. $7 + 5^x = 89$
18. $2^x - x = 7$

Miscellaneous Exercise One.

This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Copy and complete the following table of values for the general cubic

$$y = ax^3 + bx^2 + cx + d.$$

x	0	1	2	3	4
y		a + b + c + d			

1st Difference	a + b + c			
2nd Difference				
3rd Difference				

Hence determine the equation for the function having the following table of values:

x	0	1	2	3	4	5	6
y	7	7	5	-5	−29	−73	−143

2. Re-write each of the following sentences with the number written in standard form, or scientific notation, written "long hand".
   (a) Australia has an area of approximately $7{\cdot}682 \times 10^6$ km^2.
   (b) Light travels at a speed of 3×10^8 m/sec.
   (c) A golf ball has a mass of approximately $4{\cdot}5 \times 10^{-2}$ kg.
   (d) The earth is approximately $1{\cdot}5 \times 10^8$ km from the sun.
   (e) Gamma waves have a wave length less than 10^{-11} metres.
   (f) The earth orbits the sun at a speed of approximately $1{\cdot}07 \times 10^5$ km/hr.
   (g) In 1961 the first man in space, Yuri Gagarin, flew his spacecraft at a speed of $2{\cdot}74 \times 10^4$ km/hr, i.e. approximately $7{\cdot}6 \times 10^3$ m/sec.

3. Re-write each of the following sentences with the "long hand" number written in standard form, or scientific notation, i.e. in the form $A \times 10^n$ where A is a number between 1 and 10 and n is an integer.
   (a) At the beginning of this century China had a population of approximately 1 270 000 000 and India had a population of approximately 1 030 000 000.
   (b) The egg cell, or ovum, with a radius of approximately 0·000 05 metres, i.e. 0·05 mm, is the largest single human cell.
   (c) It is thought that approximately 1 100 000 people die each year of Malaria.
   (d) Some adult wasps of a particular species could weigh just 0·005 grams.
   (e) Concorde, the first supersonic passenger airliner, had a cruising speed of 2160 km/hr.

4. Round each of the following to the stated number of significant figures.

(a)	12 432 000	to	2 significant figures.
(b)	46 790	to	3 significant figures.
(c)	304 702 125	to	3 significant figures.
(d)	0·012 04	to	1 significant figures.
(e)	0·205 701	to	3 significant figures.
(f)	0·005 607	to	1 significant figures.

5. Express each of the following in the form 5^k.

(a) 25 (b) 125 (c) $\sqrt{5}$

(d) $\frac{1}{5}$ (e) $\frac{1}{25}$ (f) $\frac{1}{\sqrt{5}}$

(g) $5^3 \times 5^4$ (h) $5 \times 5^4 \times 25$ (i) $5^8 \div 5^2$

(j) $(5^3)^4$ (k) $(25)^3$ (l) 1

(m) $\frac{5^7 \times 5^1}{5^3}$ (n) $\frac{5^9}{5^3 \times 5^4}$ (o) $\frac{5^7}{15 + 10}$

6. Solve each of the following equations.

(a) $2^x = 32$ (b) $5^x = 625$ (c) $3^x = \frac{1}{9}$

(d) $(2^2)^x = 32$ (e) $8^x = 32$ (f) $64^x = 4$

(g) $125^x = \frac{1}{5}$ (h) $5^x = \frac{1}{125}$ (i) $2^{5x} = \frac{1}{4}$

(j) $25^x = \sqrt{5}$ (k) $49^x = \frac{1}{343}$ (l) $3^x = \frac{3^{10}}{9}$

7. Solve each of the following equations.

(a) $y^{-2} = 9$ (b) $p^{\frac{1}{2}} = \frac{2}{3}$ (c) $x^{-1} = \frac{2}{3}$

(d) $x^{\frac{1}{3}} = 2$ (e) $x^2 = 25$ (f) $t^{-2} = 25$

(g) $(2t)^{\frac{1}{3}} = 3$ (h) $3x^{\frac{1}{2}} + 2x^{\frac{1}{2}} = 15$ (i) $x^3 = x^2$

(j) $x^3 = x$

8. Solve the following equations:

(a) $(2^x)^2 - 10 \times 2^x + 16 = 0$ (b) $3^{2x} - 10 \times 3^x + 3^2 = 0$

(c) $2^{2x+1} - 3 \times 2^x + 1 = 0$ (d) $2^{2x-1} - 5 \times 2^x + 8 = 0$